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SOME GRACEFUL LABELINGS ON EXTENDED DUPLICATE GRAPH OF COMB GRAPH

K. Sutha[1], Dr. K. Thirusangu[1], and S. Bala[1]

[1] Department of Mathematics, S.I.V.E.T. College, Gowrivakkam, Chennai – 600 073.

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ABSTRACT

A graph labeling is a mapping that carries a set of graph elements onto a set of numbers called labels (usually the set of integers). In this paper we prove the existence of graph labeling such as Graceful, odd graceful, even graceful and (k,d) graceful labeling for extended duplicate graph of comb graph by presenting algorithms.

INTRODUCTION

Rosa introduced the concept of β valuation[6]. In 1972, Golomb called such labeling as graceful labelling[4]. Sethuraman and Elumalai have shown that $K_{1,m,n}$ with a pendent edge attached to each vertex is graceful[8]. Amutha and Kathiresan[2] proved that the graph obtained by attaching a pendent edge to each vertex $K_n + 2K_2$ is graceful. Ropp and Gallian have conjectured that every graph obtained by adding a single pendent edge to one or more vertices of a cycle is graceful[5].

The concept of odd graceful labeling was introduced by Gnanajothi in 1991[3]. She proved that the graph C_m K_2 is odd graceful if and only if m is even and every graph with an odd cycle is not odd graceful. Acharya and Hegde introduced the concept of (k,d) graceful labelling[1].

Thirusangu et al., introduced the concept of Extended Duplicate graph[7]. They proved that the Extended duplicate graph of twig graph admits graceful labeling, odd graceful, even graceful and (k,d) graceful labeling.

Thirusangu et.al,[9] proved some results on Extended Duplicate graph of Comb graph.

Definition: 1.1

Let P_{m+1} be a path graph. **Comb graph** is defined as $P_{m+1} \odot (m+1) K_1$. It has 2m+2 vertices and 2m+1 edges.

Definition: 1.2

Let G (V,E) be a Comb graph. A **Duplicate graph** of G is DG = (V_1,E_1) where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \varphi$ and f: V→V' is bijective (for v ∈ V, we write f(v) = v' for convenience) and the edge set E_1 of DG is defined as follows: The edge v_iv_j is in E if and only if both v_iv_j' and $v_i'v_j$ are edges in E_1. The **Extended Duplicate Graph** EDG of G is the graph DG $\cup\{v_iv_i'\}$, for some `i'.

Definition: 1.3

A function f is said to be graceful on a graph G with q edges if f: V → {0,1,2,....,q} such that each edge v_iv_j is assigned the label $|f(v_i) - f(v_j)|$, and the resulting edge labels are distinct numbers {1,2,3,....,q}.

Definition: 1.4

A function f is said to be odd graceful on a graph G with q edges if f: V → {0,1,2,....,2q - 1} such that each edge v_iv_j is assigned the label $|f(v_i) - f(v_j)|$, and the resulting edge labels are distinct odd numbers {1,3,5,....,2q-1}.

Definition: 1.5

A function f is said to be odd graceful on a graph G with q edges if f: V → {0,1,2,....,2q} such that each edge v_iv_j is assigned the label $|f(v_i) - f(v_j)|$, and the resulting edge labels are distinct even numbers {2,4,6,....,2q}.

Definition: 1.6

A function f is said to be (k,d) graceful on a graph G(V,E) with q edges if f : V → {0,1,2,....,k+(q -1)d} such that each edge xy is assigned the label $|f(x) - f(y)|$,and the resulting edge labels are distinct numbers {k, k+d, k+2d,...., k+(q -1)d}, for all xy ∈ E.

MAIN RESULT

In this section we present the structure of the extended duplicate graph of a comb graph and establish the existence of graceful, odd graceful, even graceful and (k,d) graceful labeling by presenting algorithms.

2.1 Definition: (Structure of the extended duplicate graph of a comb graph)

Let G (V,E) be a comb graph. DG (comb) = (V_1,E_1) is the duplicate graph of comb graph with 4m+4 vertices and 4m+2 edges.

Denote the vertex set as $V_1 = \{v_1,v_2,.........v_{2m+2},v_1',v_2'..........v_{2m+2}'\}$ and the edge set as $E_1 = \{(v_iv_{i+1}' \cup v_i'v_{i+1}$ for $1 \le i \le m) \cup (v_i'v_{m+i+1} \cup v_iv_{m+i+1}'$ for $1 \le i \le m+1)\}$. Clearly DG (comb) is disconnected. The **extended duplicate graph** of a comb EDG (Comb) is obtained from DG (comb) by adding the edges (i) v_1v_{m+1} and $v_{m+2}'v_{2m+2}'$ if $m \equiv 1 \pmod 2$. (ii) v_1v_{m+1}' and $v_{m+2}'v_{2m+2}$ if $m \equiv 0 \pmod 2$.

Thus the extended duplicate graph of comb graph has 4m+4 vertices and 4m +4 edges.

Algorithm: 1

Procedure: (Graceful labeling for EDG (comb) graph)

// assignment of labels to the vertices

If m ≡ 1(mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 2 \\ 3m-i+4 & \text{otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 4m-i+5 & \text{if } i \equiv 0 \pmod 2 \\ m+i+1 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 3m-i+4 & \text{if } i \equiv 0 \pmod 2 \\ i-1 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 4m-i+5 & \text{if } i \equiv 0 \pmod 2 \\ m+i+1 & \text{otherwise.} \end{cases}$$

}

end for

If m ≡ 0 (mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 2 \\ m+i & \text{otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 3m-i+3 & \text{if } i \equiv 0 \pmod 2 \\ 4m-i+5 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} m+i & \text{if } i \equiv 0 \pmod 2 \\ i-1 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 4m-i+5 & \text{if } i \equiv 0 \pmod 2 \\ 3m-i+3 & \text{otherwise.} \end{cases}$$

}

end for

Theorem: 2.1

The Extended duplicate graph EDG (comb) graph admits Graceful labeling.

Proof:

The EDG (comb) (V,E) graph has $p = 4m+4$ vertices and $q = 4m+4$ edges. The vertices of EDG (comb) graph are labeled by defining a function $f: V \rightarrow \{0,1,2,\ldots\ldots.q\}$ as given in algorithm 1. The induced function is defined by $f^*: E \rightarrow N$ such that $f^*(v_iv_j) = |f(v_i) - f(v_j)|$.

The induced function yields the labels for edges as follows:

for i = 1 to m ,

$$f^*(v_iv_{i+1}') = \begin{cases} 2m - 2i + 2, & \text{if } i \equiv 1 \pmod 2 \\ 4m - 2i + 5, & \text{otherwise} \end{cases}$$

$$f^*(v_i'v_{i+1}) = \begin{cases} 4m - 2i + 5, & \text{if } i \equiv 1 \pmod 2 \\ 2m - 2i + 2, & \text{otherwise} \end{cases}$$

for i = 1 to m+1

$$f^*(v_iv_{m+i+1}') = \begin{cases} 2m - 2i + 3, & \text{if } i \equiv 1 \pmod 2 \\ 4m - 2i + 6, & \text{otherwise} \end{cases}$$

$$f^*(v_i'v_{m+i+1}) = \begin{cases} 4m - 2i + 6, & \text{if } i \equiv 1 \pmod 2 \\ 2m - 2i + 3, & \text{otherwise} \end{cases}$$

If $m \equiv 1 \pmod 2$

$v_1v_{m+1} = 2m + 3$, $v_{m+2}'v_{2m+2}' = 2m+2$

If $m \equiv 0 \pmod 2$

$v_1v_{m+1}' = 2m + 3$, $v_{m+2}'v_{2m+2} = 2m+2$

Thus in both the above two cases, all the edge labels are distinct and satisfies the required condition.

Hence the Extended duplicate graph of Comb graph is Graceful.

Example: 2.1

Graceful labeling of EDG (comb) graph for m = 5 and m = 6 are given in figure 1 and figure 2 respectively.

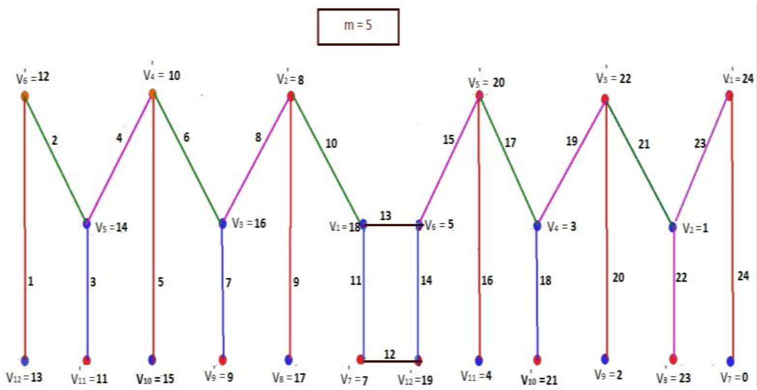

Fig. 1

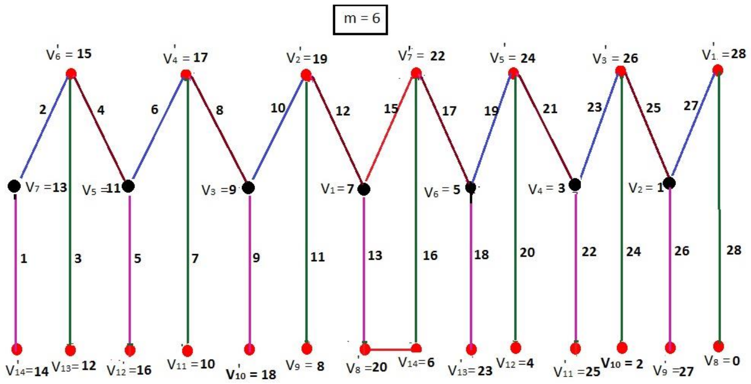

Fig. 2

Algorithm: 2

Procedure: (Odd Graceful labeling for EDG (comb) graph)

// assignment of labels to the vertices

If m ≡ 1(mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} 2i - 2 & \text{if } i \equiv 0 (\text{mod } 2) \\ 6m - 2i + 7 & \text{otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 2m + 2i + 2 & \text{if } i \equiv 0 (\text{mod } 2) \\ 8m - 2i + 9 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 6m - 2i + 7 & \text{if } i \equiv 0 (\text{mod } 2) \\ 2i - 2 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 8m - 2i + 9 & \text{if } i \equiv 0 (\text{mod } 2) \\ 2m + 2i + 2 & \text{otherwise.} \end{cases}$$

}

end for

If m ≡ 0 (mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} 2i - 2 & \text{if } i \equiv 0 (\text{mod } 2) \\ 2m + 2i & \text{otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 6m - 2i + 5 & \text{if } i \equiv 0 (\text{mod } 2) \\ 8m - 2i + 9 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 2m + 2i & \text{if } i \equiv 0 (\text{mod } 2) \\ 2i - 2 & \text{otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 8m - 2i + 9 & \text{if } i \equiv 0 \pmod 2 \\ 6m - 2i + 5 & \text{otherwise.} \end{cases}$$

}
end for
end procedure
Output: labeled vertices of EDG (comb) graph.

Theorem: 2.2

The Extended duplicate graph EDG (comb) graph admits Odd Graceful labeling.

Proof:

The EDG(comb) graph has p = 4m+4 vertices and q = 4m+4 edges. The vertices of EDG (comb) graph are labeled by defining a function f: V → {0,1,2,…,2q-1} as given in algorithm 2. The induced function is defined by f*: E → N such that $f^*(v_iv_j) = |f(v_i) - f(v_j)|$.

The induced function yields the labels for edges as follows:

for i = 1 to m+1,

$$f^*(v_iv_{m+i+1}') = \begin{cases} 4m - 4i + 5, & \text{if } i \equiv 1 \pmod 2 \\ 8m - 4i + 11, & \text{otherwise} \end{cases}$$

$$f^*(v_i'v_{m+i+1}) = \begin{cases} 8m - 4i + 11, & \text{if } i \equiv 1 \pmod 2 \\ 4m - 4i + 5, & \text{otherwise} \end{cases}$$

If m ≡ 1(mod 2)

$v_1v_{m+1} = 4m + 5$, $v_{m+2}'v_{2m+2}' = 4m+3$

for i = 1 to m

$$f^*(v_iv_{i+1}') = \begin{cases} 4m - 4i + 3, & \text{if } i \equiv 1 \pmod 2 \\ 8m - 4i + 9, & \text{otherwise} \end{cases}$$

$$f^*(v_i'v_{i+1}) = \begin{cases} 8m - 4i + 9, & \text{if } i \equiv 1 \pmod 2 \\ 4m - 4i + 3, & \text{otherwise} \end{cases}$$

If m ≡ 0(mod 2)

$v_1v_{m+1}' = 4m + 5$, $v_{m+2}'v_{2m+2} = 4m+3$

for i = 1 to m

$$f^*(v_iv_{i+1}') = \begin{cases} 8m - 4i + 9, & \text{if } i \equiv 1 \pmod 2 \\ 4m - 4i + 3, & \text{otherwise} \end{cases}$$

$$f^*(v_i'v_{i+1}) = \begin{cases} 4m - 4i + 3, & \text{if } i \equiv 1 \pmod 2 \\ 8m - 4i + 9, & \text{otherwise} \end{cases}$$

Thus in both the above two cases all the edge labels are distinct and satisfies the required condition.

Hence the Extended duplicate graph of Comb graph is Odd Graceful.

Example: 2.2

Odd graceful labeling of EDG (comb) graph for m = 5 and m = 6 are given in figure 3 and figure 4 respectively.

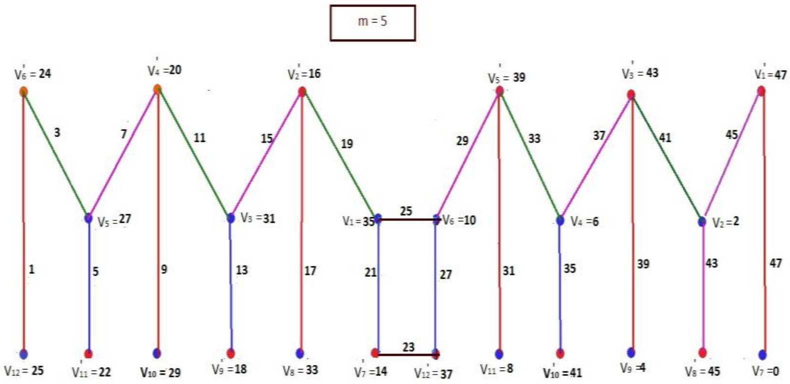

Fig. 3

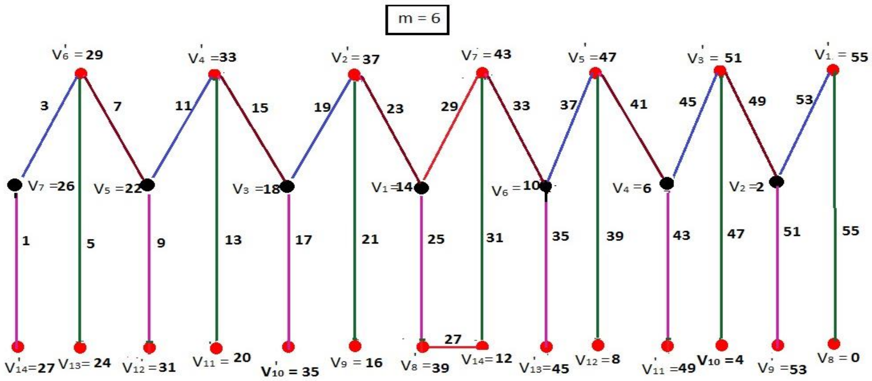

Fig. 4

Algorithm: 3

Procedure: (Even graceful labeling for EDG (comb) graph)

// assignment of labels to the vertices

If m ≡ 1 (mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} 2i - 2, \text{ if } i \equiv 0 \pmod 2 \\ 6m-2i+8, \text{ otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 2m + 2i+2, \text{ if } i \equiv 0 \pmod 2 \\ 8m - 2i+10, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 6m - 2i + 8, \text{ if } i \equiv 0 \pmod 2 \\ 2i - 2, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 8m - 2i + 10, \text{ if } i \equiv 0 \pmod 2 \\ 2m + 2i + 2, \text{ otherwise.} \end{cases}$$

}
end for

If m ≡ 0 (mod 2)
for i = 1 to m+1
{

$$v_i \leftarrow \begin{cases} 2i-2, \text{ if } i \equiv 0 \pmod 2 \\ 2m+2i, \text{ otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 6m-2i+6, \text{ if } i \equiv 0 \pmod 2 \\ 8m-2i+10, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 2m+2i, \text{ if } i \equiv 0 \pmod 2 \\ 2i-2, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} 8m-2i+10, \text{ if } i \equiv 0 \pmod 2 \\ 6m-2i+6, \text{ otherwise.} \end{cases}$$

}
end for
end procedure
Output: labeled vertices of EDG (comb) graph

Theorem: 2. 3

The Extended duplicate graph EDG (comb) graph admits Even Graceful labeling.

Proof :

The EDG (comb) (V,E) graph has p = 4m+4 vertices and q = 4m+4 edges.The vertices of EDG(comb) graph are labeled by defining a function f: V → {0,1,2,……,2q} as given in algorithm 3. The induced function is defined by f*: E → N such that $f^*(v_iv_j) = |f(v_i) - f(v_j)|$.

The induced function yields the labels for edges as follows :

for i = 1 to m,

$$f^*(v_iv_{i+1}') = \begin{cases} 4m-4i+4, \text{ if } i \equiv 1 \pmod 2 \\ 8m-4i+10, \text{ otherwise} \end{cases}$$

$$f^*(v_i'v_{i+1}) = \begin{cases} 8m-4i+10, \text{ if } i \equiv 1 \pmod 2 \\ 4m-4i+4, \text{ otherwise} \end{cases}$$

for i = 1 to m+1,

$$f^*(v_iv_{m+i+1}') = \begin{cases} 4m-4i+6, \text{ if } i \equiv 1 \pmod 2 \\ 8m-4i+12, \text{ otherwise} \end{cases}$$

$$f^*(v_i'v_{m+i+1}) = \begin{cases} 8m-4i+12, \text{ if } i \equiv 1 \pmod 2 \\ 4m-4i+6, \text{ otherwise} \end{cases}$$

If m ≡ 1 (mod 2)

$v_1v_{m+1} = 4m+6$, $v_{m+2}'v_{2m+2}' = 4m+4$

If m ≡ 0 (mod 2)

$v_1v_{m+1}' = 4m+6$, $v_{m+2}'v_{2m+2} = 4m+4$

Thus in both the above two cases all the edge labels are distinct and satisfies the required condition.

Hence the Extended duplicate graph of Comb graph is Even Graceful.

Example : 2.3

Even graceful labeling of EDG (comb) graph for m = 5 and m = 6 are given in figure 5 and figure 6 respectively.

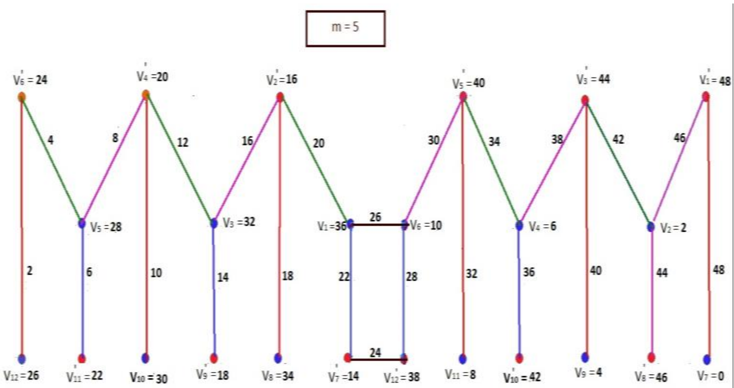

Fig. 5

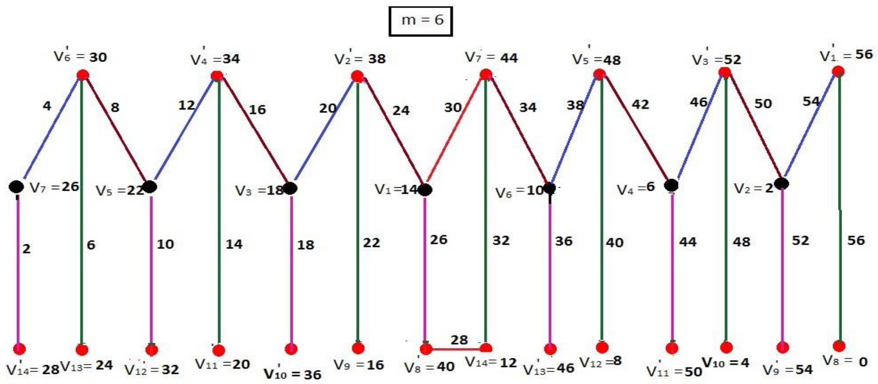

Fig. 6

Algorithm: 4

Procedure: ((k,d) Graceful labeling for EDG(comb) graph)

// assignment of labels to the vertices

If m ≡ 1 (mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} 2i-2, & \text{if } i \equiv 0 \pmod 2 \\ k+(3m-i+3)d, & \text{otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} 2m+2\,i+2, & \text{if } i \equiv 0 \pmod 2 \\ k+(4m-i+4)d, & \text{otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} k+(3m-i+3)d, \text{ if } i \equiv 0 \pmod 2 \\ 2i-2, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} k+(4m-i+4)d, \text{ if } i \equiv 0 \pmod 2 \\ 2m+2i+2, \text{ otherwise.} \end{cases}$$

}

end for

If m ≡ 0 (mod 2)

for i = 1 to m+1

{

$$v_i \leftarrow \begin{cases} 2i-2, \text{ if } i \equiv 0 \pmod 2 \\ 2m+2i, \text{ otherwise.} \end{cases}$$

$$v_i' \leftarrow \begin{cases} k+(3m-i+2)d, \text{ if } i \equiv 0 \pmod 2 \\ k+(4m-i+4)d, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1} \leftarrow \begin{cases} 2m+2i, \text{ if } i \equiv 0 \pmod 2 \\ 2i-2, \text{ otherwise.} \end{cases}$$

$$v_{m+i+1}' \leftarrow \begin{cases} k+(4m-i+4)d, \text{ if } i \equiv 0 \pmod 2 \\ k+(3m-i+2)d, \text{ otherwise.} \end{cases}$$

}

end for

Theorem: 2. 4

The Extended duplicate graph EDG (comb) graph admits (k,d) Graceful labeling.

Proof:

The EDG (comb) (V,E) graph has p = 4m+4 vertices and q = 4m+4 edges. The vertices of EDG(comb) graph are labeled by defining a function f: V → {0,1,2,…….,k+(q-1)d} as given in algorithm 4. The induced function is defined by f*: E → N such that $f^*(v_iv_j) = |f(v_i) - f(v_j)|$.

The induced function yields the labels for edges as follows:

for i = 1 to m

$$f^*(v_iv_{i+1}') = \begin{cases} k+(2m-2i+1)d, \text{ if } i \equiv 1 \pmod 2 \\ k+(4m-2i+4)d, \text{ otherwise} \end{cases}$$

$$f^*(v_i'v_{i+1}) = \begin{cases} k+(4m-2i+4)d, \text{ if } i \equiv 1 \pmod 2 \\ k+(2m-2i+1)d, \text{ otherwise} \end{cases}$$

for i = 1 to m+1

$$f^*(v_iv_{m+i+1}') = \begin{cases} k+(2m-2i+2)d \text{ if } i \equiv 1 \pmod 2 \\ k+(4m-2i+5)d \text{ otherwise} \end{cases}$$

$$f^*(v_i'v_{m+i+1}) = \begin{cases} k+(4m-2i+5)d & \text{if } i \equiv 1 \pmod 2 \\ k+(2m-2i+2)d & \text{otherwise} \end{cases}$$

If m ≡ 1 (mod 2)

$v_1v_{m+1} = k+4m+4$, $v_{m+2}'v_{2m+2}' = k+4m+2$

If m ≡ 0 (mod 2)

$v_1v_{m+1}' = k+4m+4$, $v_{m+2}'v_{2m+2} = k+4m+2$

Thus in both the above two cases, the edge labels are {k, k+d, k+2d,……,k+(4m+3)d} which are distinct and satisfies the required condition.

Hence the Extended duplicate graph of Comb graph is (k,d) Graceful.

Example: 2.4

(k,d) graceful labeling of EDG (comb) graph for m = 5 and m = 6 are given in figure 7 and figure 8 respectively.

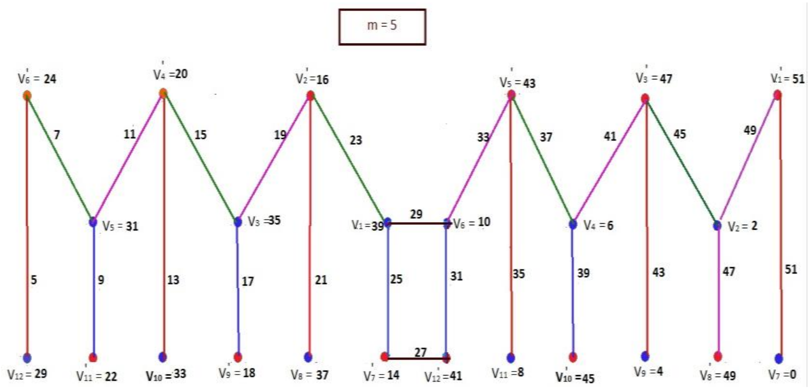

Fig. 7

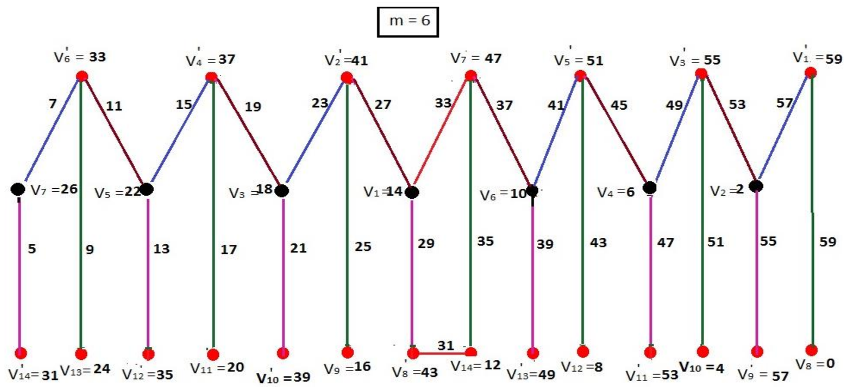

Fig. 8

CONCLUSION

We proved the existence of graceful labeling, odd graceful, even graceful and (k,d) graceful labelings for the Extended Duplicate graph of comb graph.

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